

A finite element exterior calculus framework for the rotating shallow-water equations

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Abstract

We describe discretisations of the shallow water equations on the sphere using the framework of finite element exterior calculus, which are extensions of the mimetic finite difference framework presented in (Ringler, Thuburn, Klemp, and Skamarock, Journal of Computational Physics (2010)). We present two formulations, a “primal” formulation in which the finite element spaces are defined on a single mesh, and a “primal-dual” formulation in which a finite element spaces on a dual mesh are also used. Both formulations have velocity and layer depth as prognostic variables, but the exterior calculus framework leads to a conserved diagnostic potential vorticity. In both formulations we show how to construct discretisations that have mass-consistent (constant potential vorticity stays constant), stable and oscillation free potential vorticity advection.

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1. Introduction

In a recent paper on horizontal grids for global weather and climate models, [ST12] listed a number of desirable properties that a numerical discretisation should have, which can be paraphrased as accurate representation of geostrophic adjustment, mass conservation, curl-free pressure gradient, energy conserving pressure terms, energy conserving Coriolis term, steady geostrophic modes, and absence/control of spurious modes. Of this list as presented here, the first property could be said to relate to the stability and accuracy of the discrete Laplacian formed from divergence and gradient operators, whilst the next five all relate to mimetic properties (*i.e.* the numerical discretisations exactly represent differential calculus identities such as $\nabla \times \nabla = 0$), and the last property relates to the kernels of the various discretised operators (see [LSRH05, LRP07, LP08] and related papers by Le Roux and coworkers for extended discussion of these issues in the context of finite element methods). In the context of the rotating shallow-water equations on the sphere, which represent the standard nonlinear framework for investigating horizontal grids for global models, the C-grid staggering on the latitude-longitude grid combined with an appropriate choice of reconstruction of the Coriolis term provides all of these properties, but leaves us with a grid system with a polar singularity. This, together with a need for models with variable resolution, has started a quest for alternative grids and discretisations that satisfy these properties.

The extension of the C-grid to triangular meshes (and the finite element analogue, the RT0-P0 discretisation) satisfies the first six properties and has been popular in both atmosphere and ocean applications ([WC98, BR05]), however it is now well understood that the triangular C-grid supports spurious inertia-gravity mode branches because of the decreased ratio of velocity degrees of freedom (DOFs) to pressure DOFs relative to quadrilaterals (from 2:1 to 3:2) [Dan10, Gas11]. More recently, a Coriolis reconstruction for the hexagonal C-grid was derived in [Thu08] that provides the mimetic properties described above, and this was extended to arbitrary orthogonal C-grids (grids in which dual grid edges that join pressure points intersect the primal grid edges orthogonally) in [TRSK09]. The hexagonal C-grid has an increased ratio of velocity DOFs to pressure DOFs (from 2:1 to 3:1), and so does not support spurious inertia-gravity mode branches, but does have a branch of spurious Rossby modes. This reconstruction can be used to construct energy and enstrophy conserving C-grid discretisations for the nonlinear rotating shallow-water equations using the vector invariant form [RTKS10], in which mimetic properties are used to produce a velocity-pressure formulation in which the diagnosed potential vorticity is locally conserved, is consistent with the mass conservation (*i.e.* constant potential vorticity stays constant in the unforced case), and can be controlled such that it remains bounded.

Two directions remain outstanding from this approach, namely the relaxation of the orthogonality requirement which constrains cubed sphere grids so that grid resolution increases much more quickly in the corners than at the middle of the faces [PL07], and the construction of higher-order operators to avoid grid imprinting. Two recent papers by the authors attempted to address these issues. In [TC12] a framework was set up to generalise the mimetic approach of [RTKS10] to non-orthogonal grids, but the method of constructing sufficiently high-order operators was not clear. Meanwhile, in [CS12], it was shown that mixed finite element methods in the framework of finite element exterior calculus (see [AFW06] for a review) provide the first six properties listed above, plus sufficient flexibility to adjust the ratio of velocity DOFs to pressure DOFs to 2:1 to avoid spurious mode branches. The BDFM1 space on triangles and the RTk hierarchy of spaces on quadrilaterals were advocated as examples of spaces that satisfy that ratio. However, in that paper it was not clear how the extension to nonlinear shallow-water equations would be made. In this paper we address both of these open questions by describing a finite element exterior calculus framework for the shallow-water equations, which enables us to write the equations in a very compact form and reveals the underlying structure behind the mimetic properties. We shall discuss two formulations, a primal grid formulation in which potential vorticity is represented on a continuous finite element space, and a primal-dual grid formulation that makes use of the discrete Hodge star operator introduced in [Hip01a, Hip01b], in which potential vorticity is represented on a discontinuous finite element space. In the latter case, discontinuous Galerkin or finite volume methods can be used for locally conservative, bounded, mass-consistent potential vorticity advection, whilst in the former case we show that streamline-upwind Petrov-Galerkin methods with discontinuity-capturing schemes can be incorporated into the framework to provide conservative, high-order, stable, non-oscillatory advection of potential vorticity.

The rest of this paper is structured as follows. In Section 2 we provide a “hands-on” introduction to the calculus of differential forms, then write the rotating shallow water equations in differential form notation. In Section 3.2 we describe our primal grid finite element exterior calculus formulation of the shallow water equations, and in Section 4 we describe

our primal/dual grid formulation. Finally, in Section 5, we provide a summary and outlook.

2. Differential forms on manifolds

In this section, we introduce the required elements from the language of differential forms, in an informal manner. For more rigorous definitions, the reader is referred to [MR99, AFW06, Hol11]. We then combine these elements to write the rotating shallow-water equations on the sphere in differential form notation.

2.1. Differential form preliminaries

Solution domain. We shall consider the case in which the solution domain Ω is a closed compact two-dimensional surface embedded in three dimensional space. In applications the main surfaces of interest are the surface of the sphere, or a rectangle in the $x - y$ plane with periodic boundary conditions in both Cartesian directions. For brevity of notation we do not consider domains with boundaries; this avoids the need to include boundary terms when integrating by parts, although they can easily be included. We shall expand all quantities in three-dimensional Cartesian coordinates, and shall use the machinery of contraction operators (defined later) to restrict the equations to the two-dimensional surface Ω . Hence, we begin by developing differential calculus in \mathbb{R}^3 .

Vector fields. Informally, a vector field \mathbf{u} on \mathbb{R}^3 is a mapping $\mathbf{u} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, *i.e.* a vector field assigns arbitrary 3-vectors to each point in \mathbb{R}^3 . We denote $\mathfrak{X}(\mathbb{R}^3)$ as the space of vector fields tangent to Lagrangian flow lines on \mathbb{R}^3 . Physically, velocity fields are represented as vector fields. In due course we shall also require vector fields on Ω , which are only defined for points $\mathbf{x} \in \Omega$, and the assigned vectors are constrained to be tangential to Ω . We denote $\mathfrak{X}(\Omega)$ as the space of vector fields on Ω .

Differential forms. In this paper we shall make use of four types of differential forms: 0-forms (which are simply scalar-valued functions), 1-forms, 2-forms, and 3-forms. In general, 1-forms are used to compute line integrals, 2-forms are used to compute surface integrals, and 3-forms are used to compute volume integrals. We shall denote Λ^k as the space of k -forms.

Cotangent vectors on \mathbb{R}^3 . To define differential forms, we first need to define cotangent vectors which are the dual objects to vectors, *i.e.* mappings from tangent vectors to \mathbb{R} . In \mathbb{R}^3 , cotangent vectors have the basis $\{dx^i\}_{i=1}^3$ which is dual to the Cartesian basis $\{\hat{\mathbf{x}}^i\}_{i=1}^3$ for tangent vectors, so the general form for cotangent vectors is

$$m = \sum_i m_i dx^i \equiv \mathbf{m} \cdot d\mathbf{x}, \quad \mathbf{m} = (m_1, m_2, m_3)^T, \quad d\mathbf{x} = (dx^1, dx^2, dx^3)^T.$$

Cotangent vectors \mathbf{m} are defined by their result when the inner product is taken with tangent vectors \mathbf{v} , *i.e.* $\langle \mathbf{m} \cdot d\mathbf{x}, \mathbf{v} \rangle = \mathbf{m}^T \mathbf{v}$.

1-forms on \mathbb{R}^3 . A differential 1-form ω assigns a cotangent vector to each point $\mathbf{x} \in \mathbb{R}^3$. The inner product $\langle \omega, \mathbf{u} \rangle$ of a 1-form ω and a vector field \mathbf{u} is therefore a scalar function (0-form) defined on \mathbb{R}^3 . On \mathbb{R}^3 , it is convenient to expand 1-forms in the standard basis for three-dimensional cotangent vectors, *i.e.*, (dx^1, dx^2, dx^3) . Informally, a 1-form can be integrated along a one-dimensional curve $C \subset \mathbb{R}^3$ using the usual definition of line integration

$$\int_C \omega = \int_C \sum_i \omega_i dx^i.$$

Wedge product and volume forms. The wedge product can be used to construct 2-forms and 3-forms from 1-forms. The wedge product of a scalar function (0-form) f with a k -form ω is simply the arithmetic product:

$$f \wedge \omega = f\omega.$$

In general, the wedge product of an n -form and an m -form produces an $(n+m)$ -form. The wedge product satisfies the following conditions:

1. Bilinearity:

$$(a\alpha_1 + b\alpha_2) \wedge \omega = a(\alpha_1 \wedge \omega) + b(\alpha_2 \wedge \omega), \quad \alpha \wedge (a\omega_1 + b\omega_2) = a(\alpha \wedge \omega_1) + b(\alpha \wedge \omega_2),$$

2. Anticommutativity:

$$\omega \wedge \gamma = (-1)^{kl} \gamma \wedge \omega,$$

for a k -form ω and an l -form γ , and

3. Associativity:

$$(\omega \wedge \gamma) \wedge \kappa = \omega \wedge (\gamma \wedge \kappa).$$

From these properties it may be deduced that the wedge product of two arbitrary 1-forms ω, γ takes the form

$$\omega \wedge \gamma = \alpha_1 dx^1 \wedge dx^2 + \alpha_2 dx^1 \wedge dx^3 + \alpha_3 dx^2 \wedge dx^3,$$

for some scalar functions $\alpha_1, \alpha_2, \alpha_3$, and hence this is the general form for 2-forms in Cartesian coordinates. Similarly it may be deduced that the wedge product of three arbitrary 1-forms ω, γ and κ takes the form

$$\omega \wedge \gamma \wedge \kappa = f(\mathbf{x}) dx^1 \wedge dx^2 \wedge dx^3,$$

for some scalar function $f(\mathbf{x})$, and hence this is the general structure of 3-forms. We define the “volume form”

$$dV = dx^1 \wedge dx^2 \wedge dx^3.$$

A general 3-form $f(\mathbf{x}) dV$ may be integrated over three-dimensional submanifolds $M \in \mathbb{R}^3$ using the usual definition of triple integration:

$$\int_M f dV = \int_M f dx^1 dx^2 dx^3.$$

Contraction with vector fields. Having defined the volume form, we can use contraction with vector fields to construct 1-forms and 2-forms on the surface Ω . In general, contraction of a vector field \mathbf{u} with a k -form ω results in a $(k-1)$ -form, denoted $\mathbf{u} \lrcorner \omega$. The contraction of a vector field \mathbf{u} with a 1-form ω is simply the inner product

$$\mathbf{u} \lrcorner \omega = \langle \mathbf{u}, \omega \rangle = \sum_i u^i \omega_i.$$

The contraction is linear, *i.e.* $\mathbf{u} \lrcorner (a(x)\omega_1 + b(x)\omega_2) = a(x)\mathbf{u} \lrcorner \omega_1 + b(x)\mathbf{u} \lrcorner \omega_2$, for two scalar functions $a(x)$ and $b(x)$, and two k -forms ω_1 and ω_2 . Hence, we may define the contraction with 2-forms for the basis elements $dx^1 \wedge dx^2$, $dx^1 \wedge dx^3$, and $dx^2 \wedge dx^3$, with

$$\mathbf{u} \lrcorner dx^i \wedge dx^j = u_i dx^j - u_j dx^i, \quad 1 \leq i \neq j \leq 3.$$

Similarly, the contraction with 3-forms can be defined from

$$\mathbf{u} \lrcorner dx^1 \wedge dx^2 \wedge dx^3 = u_1 dx^2 \wedge dx^3 - u_2 dx^1 \wedge dx^3 + u_3 dx^1 \wedge dx^2.$$

Surface form on Ω . The surface form dS is used for surface integrals over two-dimensional submanifolds M of Ω ; it is defined on Ω as $dS = \hat{\mathbf{k}} \lrcorner dV$ where $\hat{\mathbf{k}}$ is the unit oriented vector field normal to Ω . The general form of 2-forms on Ω is then

$$\omega = \rho(\mathbf{x}) dS,$$

for scalar functions $\rho(\mathbf{x})$ defined on Ω . A 2-form on Ω can be integrated over a two dimensional submanifold M of Ω using the usual definition of surface integration:

$$\int_M \omega = \int_M \rho(\mathbf{x}) dS.$$

Identification of vector fields on Ω with 1-forms. In this framework we shall make use of two different identifications of vector fields on Ω with 1-forms. The first is written

$$\mathbf{u} \mapsto \mathbf{u} \cdot d\mathbf{x}_\Omega = \sum_i u_i dx_\Omega^i,$$

where

$$d\mathbf{x}_\Omega = d\mathbf{x} - \hat{\mathbf{k}}(\hat{\mathbf{k}} \cdot d\mathbf{x}),$$

i.e. the projection of $d\mathbf{x}$ into the dual space to the tangent plane on Ω at \mathbf{x} . This identification is used to compute circulation integrals

$$\int_C \mathbf{u} \cdot d\mathbf{x}_\Omega \equiv \int_C \mathbf{u} \cdot d\mathbf{x},$$

along curves $C \subset \Omega$, and hence is associated with the curl operator on Ω . The second identification is written using the contraction with dS ,

$$\mathbf{u} \mapsto \mathbf{u} \lrcorner dS,$$

and is used to compute flux integrals

$$\int_C \mathbf{u} \lrcorner dS = \int_C \mathbf{u} \cdot \hat{\mathbf{k}} \times d\mathbf{x}$$

across curves $C \subset \Omega$, and hence is associated with the divergence operator on Ω .

We shall denote $\Lambda^0(\Omega)$ as the space of 0-forms (scalar functions) on Ω , $\Lambda^1(\Omega)$ as the space of 1-forms on Ω , which can be written in both of the identifications with vectors fields given above, and $\Lambda^2(\Omega)$ as the space of 2-forms on Ω written in the form $\omega = \rho(\mathbf{x}) dS$ where ρ is a scalar function.

Differential operator. The differential operator d neatly encodes all of the vector calculus differential operators *e.g.*, div , grad , curl *etc.* In general, the differential operator d maps k -forms to $k + 1$ -forms, and satisfies:

1. For scalar functions f , $d f = \sum_i \frac{\partial f}{\partial x^i} d x^i = \nabla f \cdot d \mathbf{x}$.
2. Product rule: $d(\omega \wedge \gamma) = (d \omega) \wedge \gamma + \omega \wedge (d \gamma)$.
3. Closure: $\iff d(d \omega) = d^2 \omega = 0$.

We shall see some examples very shortly, however, first we shall write the general form of Stoke's theorem for d :

$$\int_M d \omega = \int_{\partial M} \omega,$$

where M is a k -dimensional submanifold of \mathbb{R}^3 , ω is a $(k-1)$ -form (and hence $d \omega$ is a k -form), and ∂M is the $(k-1)$ -dimensional submanifold of \mathbb{R}^3 corresponding to the boundary of M . Combining Stoke's theorem with the product rule provides the integration by parts formula

$$\int_M (d \omega) \wedge \gamma = \int_{\partial M} \omega \wedge \gamma - \int_M \omega \wedge (d \gamma).$$

Differential operators on Ω . Standard vector calculus differential operators on scalar functions f and vectors fields \mathbf{u} defined on Ω are obtained from the two identifications of vector fields with 1-forms:

$$\begin{aligned} d f &= \nabla f \cdot d \mathbf{x}_\Omega, \\ d f &= (\hat{\mathbf{k}} \times \nabla f) \lrcorner d S := \nabla^\perp f \lrcorner d S, \\ d(\mathbf{u} \cdot d \mathbf{x}_\Omega) &= \hat{\mathbf{k}} \cdot \nabla \times \mathbf{u} d S := \nabla^\perp \cdot \mathbf{u} d S, \\ d(\mathbf{u} \lrcorner d S) &= \nabla \cdot \mathbf{u} d S, \end{aligned}$$

where ∇ , ∇^\perp , $\nabla^\perp \cdot$ and $\nabla \cdot$ are differential operators defined intrinsically on the two-dimensional surface Ω . The closure property $d^2 = 0$ then leads to the following vector identities for the two identifications of vector fields with 1-forms:

$$0 = d^2 f = d(\nabla f \cdot d \mathbf{x}_\Omega) = \nabla^\perp \cdot \nabla f d S, \quad (1)$$

$$0 = d^2 f = d(\nabla^\perp f \lrcorner d S) = \nabla \cdot \nabla^\perp f d S. \quad (2)$$

These identities are crucial for geophysical applications since they dictate the scale separation between slow and fast dynamics.

Hodge star on Ω . The Hodge star operator \star on Ω maps from k -forms to $(2-k)$ -forms. It is used in this paper to define the L_2 -inner product between two k -forms ω and γ by

$$\langle \omega, \gamma \rangle = \int_\Omega \omega \wedge \star \gamma,$$

and is also used to define the Coriolis term. The Hodge star is linear (*i.e.*, $\star(a(x)\omega + b(x)\gamma) = a(x)\star\omega + b(x)\star\gamma$ for scalar functions a, b and k -forms ω, γ), and so we may define it for our chosen Cartesian basis, where the Hodge star satisfies

$$\begin{aligned} \star f &= f d S, \\ \star f d S &= f, \\ \star(\mathbf{u} \cdot d \mathbf{x}_\Omega) &= \mathbf{u} \lrcorner d S = \hat{\mathbf{k}} \times \mathbf{u} \cdot d \mathbf{x}_\Omega, \\ \star \mathbf{u} \lrcorner d S &= -(\mathbf{u} \cdot d \mathbf{x}_\Omega) = \hat{\mathbf{k}} \times \mathbf{u} \lrcorner d S. \end{aligned}$$

From the presence of $\hat{\mathbf{k}} \times$ in these formulas it becomes clear why the Hodge star might be relevant to the Coriolis term. Note that $\star \star = \text{Id}$ for 0- and 2-forms, and $\star \star = -\text{Id}$ for 1-forms.

2.2. Rotating shallow-water equations in differential form notation

We have now established enough notation to write the rotating shallow-water equations on Ω in differential form notation. We start from the following form of the rotating shallow-water equations:

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u} + (\zeta + f) \mathbf{u}^\perp + \nabla(g(D - b) + \frac{1}{2}|\mathbf{u}|^2) &= 0, \\ \frac{\partial}{\partial t} D + \nabla \cdot (\mathbf{u} D) &= 0, \end{aligned}$$

where \mathbf{u} is the velocity, $\zeta = \hat{\mathbf{k}} \cdot \nabla \times \mathbf{u} = \nabla^\perp \cdot \mathbf{u}$ is the vorticity, D is the layer depth, b is the height of the bottom surface, g is the acceleration due to gravity, f is the Coriolis parameter and $^\perp = \hat{\mathbf{k}} \times$.

Using the notation that we have described above, we can immediately rewrite these equations as

$$\frac{\partial}{\partial t} \mathbf{u} \cdot d\mathbf{x}_\Omega + \star((\zeta + f)\mathbf{u} \cdot d\mathbf{x}_\Omega) + d\left(g(D - b) + \frac{1}{2}|\mathbf{u}|^2\right) = 0, \quad (3)$$

$$\frac{\partial}{\partial t} D dS + d(\mathbf{u} \lrcorner D dS) = 0, \quad (4)$$

$$d(\mathbf{u} \cdot d\mathbf{x}_\Omega) = \zeta dS. \quad (5)$$

Applying d to Equation (3) and making use of $d^2 = 0$ and the definition of Hodge star immediately gives the vorticity equation

$$\frac{\partial}{\partial t} \zeta dS + d(\mathbf{u} \lrcorner (\zeta + f) dS) = 0,$$

which is in the same flux form as the mass equation (Equation (4)). The potential vorticity (PV) q is defined from

$$(\zeta + f) dS = q D dS,$$

and hence we obtain the law of conservation of potential vorticity

$$\frac{\partial}{\partial t} (q D dS) + d(\mathbf{u} \lrcorner q D dS) = 0. \quad (6)$$

Note that if q is constant then

$$\left(\frac{\partial q}{\partial t} D dS\right) + q \underbrace{\left(\frac{\partial}{\partial t} D dS + d(\mathbf{u} \lrcorner q D dS)\right)}_{=0} = 0, \quad \frac{\partial q}{\partial t} = 0,$$

which means that q remains constant. This is what is meant by consistency with Equation (4).

Our goal is to design a framework for finite element discretisations that has \mathbf{u} and D as the prognostic variables, yet preserves the conservation law structure of equations (4) and (6). Furthermore, we shall show how stabilisations for these conservation laws (which are required for meteorological applications) can be incorporated into this framework.

3. Finite element exterior calculus formulation

3.1. Finite element spaces

The fundamental idea of finite element exterior calculus applied to the rotating shallow-water equations is to choose finite element spaces for \mathbf{u} , ζ and D such that the operator d maps from one expansion to another, so that the vector calculus identities (1) and (2) still hold. The difficulty is that continuous derivatives in the normal direction across element boundaries are required to compute $d(\mathbf{u} \lrcorner dS)$, whilst continuous derivatives in the tangent direction across element boundaries are required to compute $d(\mathbf{u} \cdot d\mathbf{x}_\Omega)$. On a single grid, we cannot have both, and thus must choose to construct finite element expansions such that only one of (1) or (2) hold in the strong form, and the other will hold in the weak form having integrating by parts. This amounts to choosing one of D and ζ to have a continuous finite element expansion and the other to have a discontinuous expansion. A discontinuous expansion allows for discontinuous Galerkin methods which are locally conservative and allow for monotonic advection schemes, and in meteorological applications it is more important that these schemes are available for D than ζ , so we choose to hold (1) in the strong form. Later, in setting up the primal-dual grid formulation, we shall introduce a dual grid for which (2) holds in the strong form, consistently with the weak form on the primal grid.

Finite element differential form spaces. Having made the choice to hold (1) in strong form, we need to choose finite element spaces for ζ , \mathbf{u} , and D , denoted E , S and V respectively. This choice defines equivalent subspaces $\hat{\Lambda}^k(\Omega) \subset \Lambda^k(\Omega)$, $k = 1, 2, 3$, given by

$$\begin{aligned}\hat{\Lambda}^0(\Omega) &= E, \\ \hat{\Lambda}^1(\Omega) &= \{\mathbf{u} \lrcorner dS : \mathbf{u} \in S\}, \\ \hat{\Lambda}^2(\Omega) &= \{D \lrcorner dS : D \in V\},\end{aligned}$$

We require that d maps from $\hat{\Lambda}^1(\Omega)$ into $\hat{\Lambda}^2(\Omega)$, and that d maps from $\hat{\Lambda}^0(\Omega)$ into $\hat{\Lambda}^1(\Omega)$, which implies that E is a continuous finite element space, S is div-conforming (*i.e.* $\mathbf{u} \in S$ has continuous normal components across element boundaries), and V is a discontinuous finite element space. Numerous examples of (E, S, V) satisfying these properties, including $E = P(k)$, $S = BDM(k)$, $V = P(k)_{DG}$.

Dual differential operator. We define δ as the dual differential operator from $\hat{\Lambda}^k$ to $\hat{\Lambda}^{k-1}$ that is dual to d , *i.e.*

$$\int_{\Omega} \gamma \wedge \star \delta \omega = - \int_{\Omega} d\gamma \wedge \star \omega, \quad \forall \gamma \in \hat{\Lambda}^k.$$

We note that $\delta^2 \omega = 0$, since

$$\begin{aligned}\int_{\Omega} \gamma \wedge \star \delta^2 \omega &= - \int_{\Omega} d\gamma \wedge \star \delta \omega \\ &= \int_{\Omega} d^2 \gamma \wedge \star \omega \\ &= 0 \quad \forall \gamma \in \Lambda^0, \omega \in \Lambda^2.\end{aligned}$$

Discrete Helmholtz decomposition. As discussed in [AFW06], if d maps from $\hat{\Lambda}^0(\Omega)$ onto the kernel of d in $\hat{\Lambda}^1(\Omega)$, then there is discrete Helmholtz decomposition and any 1-form $\mathbf{u} \lrcorner dS \in \hat{\Lambda}^1(\Omega)$ can be written as

$$\mathbf{u} \lrcorner dS = d\psi + \delta\phi + \mathfrak{h},$$

where $\psi \in \hat{\Lambda}^0(\Omega)$, $\phi \in \hat{\Lambda}^2(\Omega)$, and $\mathfrak{h} \in \mathcal{H}$, where $\mathcal{H} \subset \hat{\Lambda}^1(\Omega)$ is the space of discrete harmonic 1-forms given by

$$\mathcal{H} = \{\mathfrak{h} \in \hat{\Lambda}^1(\Omega) : d\mathfrak{h} = 0, \delta\mathfrak{h} = 0\},$$

which has the same dimension as the space of continuous 1-forms on Ω (which has dimension 0 for the surface of a sphere).

Construction of global finite element spaces by pullback. We construct the spaces $\hat{\Lambda}^k(\Omega)$, $k = 0, 1, 2$, element by element, and then enforce continuity constraints, namely: $\gamma \in \hat{\Lambda}^0(\Omega)$ is fully continuous, $\mathbf{u} \lrcorner dS \in \hat{\Lambda}^1(\Omega)$ has continuous normal components across element boundaries, and $\phi dS \in \hat{\Lambda}^2(\Omega)$ is allowed to be fully discontinuous. The restriction $\hat{\Lambda}^k(e)$ to element e of $\hat{\Lambda}^k(\Omega)$ is defined through a pullback mapping $\phi_e : \hat{\Lambda}^k(\hat{e}) \rightarrow \hat{\Lambda}^k(e)$ from a reference element \hat{e} (where integrals are computed) to each element e , the pullback $\phi_e^* : \hat{\Lambda}^k(e) \rightarrow \hat{\Lambda}^k(\hat{e})$, is defined by

$$\int_M \phi_e^* \omega = \int_{\phi_e(M)} \omega,$$

for all $\omega \in \Lambda^k(e)$ and all integrable submanifolds $M \in \hat{e}$. Specifically:

$$\begin{aligned} \hat{\gamma} = \phi_e^* \gamma &= \gamma \circ \phi, & \gamma \in \hat{\Lambda}^0(e), & \hat{\gamma} \in \hat{\Lambda}^0(\hat{e}) \\ \hat{\mathbf{u}} \lrcorner d\hat{S} = \phi_e^* \mathbf{u} \lrcorner dS &= \mathbf{u} \lrcorner \mathbf{k} \lrcorner d\phi_1 \wedge d\phi_2 \wedge d\phi_3, & \mathbf{u} \lrcorner dS \in \hat{\Lambda}^0(e), & \hat{\mathbf{u}} \lrcorner d\hat{S} \in \hat{\Lambda}^0(\hat{e}), \\ &= \mathbf{u} \lrcorner d\phi, & \mathbf{u} \lrcorner dS \in \hat{\Lambda}^1(e), & \\ \hat{D} d\hat{S} = \phi_e^* D dS &= D d\hat{S}, \end{aligned}$$

where $d\hat{S} = d\hat{x} \wedge d\hat{y}$ is the surface form on \hat{e} with coordinates \hat{x} , \hat{y} , and ϕ_i , $i = 1, 2, 3$ are the components of ϕ . In coordinates, the pullback of $\mathbf{u} \lrcorner dS$ defines the Piola transformation

$$\phi^*(\mathbf{u} \lrcorner dS) = \frac{1}{\det\left(\frac{\partial\phi}{\partial\mathbf{x}}\right)} \frac{\partial\phi}{\partial\mathbf{x}} \hat{\mathbf{u}} \lrcorner d\hat{S},$$

and the pullback of $D dS$ defines the scaling transformation

$$\phi^*(D dS) = \frac{\hat{D}}{\det\left(\frac{\partial\phi}{\partial\mathbf{x}}\right)} d\hat{S}.$$

The pullback operator ϕ^* commutes with d , *i.e.* $d\phi^*\omega = \phi^*d\omega$, so it is sufficient to check that d maps from $\hat{\Lambda}^k(\hat{e})$ to $\hat{\Lambda}^{k+1}(\hat{e})$, $k = 1, 2$ to guarantee that it maps from $\hat{\Lambda}^k(\Omega)$ to $\hat{\Lambda}^{k+1}(\Omega)$.

3.2. Primal grid formulation

Here we shall concentrate on the semi-discrete continuous time equations.

Discrete velocity equation. Since we have chosen to use the strong definition of $d(\mathbf{u} \lrcorner dS)$, we need to transform equation (3) into a form where this choice of identification with vector fields is used. This is done by applying the Hodge star:

$$\frac{\partial}{\partial t} \mathbf{u} \lrcorner dS + \star(\mathbf{Q} \lrcorner dS) + \star d \star (g(D - b) + K) dS = 0, \quad \mathbf{Q} = (\zeta + f)\mathbf{u},$$

where $K dS = |\mathbf{u}|^2/2 dS$ and we have replaced $D - b + K$ with $\star(D - b + K) dS$. Note that we shall need to be careful with our definition of the PV flux \mathbf{Q} for consistency, this shall be discussed later.

Since $\hat{\Lambda}^2(\Omega)$ is a discontinuous space, we cannot apply d to $D dS$ and must adopt the weak form. This is done by taking the L_2 inner product with a test 1-form $\mathbf{w} \lrcorner dS$ using the Hodge star, and integrating by parts (with vanishing boundary term since there are no boundaries):

$$\frac{d}{dt} \int_{\Omega} \mathbf{w} \lrcorner dS \wedge \star \mathbf{u} \lrcorner dS - \int_{\Omega} \mathbf{w} \lrcorner dS \wedge \mathbf{Q} \lrcorner dS - \int_{\Omega} d(\mathbf{w} \lrcorner dS) \wedge \star (g(D - b) + K) dS = 0, \quad \forall \mathbf{w} \lrcorner dS \in \hat{\Lambda}^1(\Omega). \quad (7)$$

To obtain the Galerkin finite element approximation of this equation we simply restrict $\mathbf{u} \lrcorner dS$, $\mathbf{w} \lrcorner dS$ and $\mathbf{Q} \lrcorner dS$ to the finite element space $\hat{\Lambda}^1(\Omega)$, and $D dS$ and $b dS$ to the finite element space $\hat{\Lambda}^2(\Omega)$.

Discrete mass equation. In deriving the equation for D , the integral must be performed over a single element e due to the discontinuity, following the discontinuous Galerkin approach. Taking the L_2 inner product of a test 2-form $\phi dS \in \hat{\Lambda}^2(\Omega)$ with equation (4) over one element e gives

$$\frac{d}{dt} \int_e \phi dS \wedge \star(D ds) = - \int_e \phi dS \wedge \star d(\mathbf{u} \lrcorner D dS).$$

To obtain coupling between elements we integrate by parts to obtain

$$\frac{d}{dt} \int_e \phi dS \wedge \star(D ds) = \int_e d \star(\phi dS) \wedge \mathbf{u} \lrcorner D dS - \int_{\partial e} \star(\phi dS) \wedge \mathbf{u} \lrcorner \tilde{D} dS, \quad \forall \phi dS. \quad (8)$$

where ∂e is the boundary of element e and \tilde{D} is chosen as the value of D on the upwind side, following the standard discontinuous Galerkin approach. Again we obtain the finite element approximation to equation (4) upon restriction of ϕdS and $D dS$ to $\hat{\Lambda}^2(\Omega)$, and $\mathbf{u} \lrcorner dS$ to $\hat{\Lambda}^1(\Omega)$. In the case of $\hat{\Lambda}^2(\Omega)$ being piecewise constant functions, we may select ϕ as the indicator function for element e , *i.e.*,

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in e, \\ 0 & \text{otherwise,} \end{cases}$$

and Equation (8) becomes

$$\frac{d}{dt} \int_e D ds = - \int_{\partial e} \mathbf{u} \lrcorner \tilde{D} dS,$$

which is the usual finite volume scheme for D , in which case a higher-order upwind flux $\mathbf{u} \tilde{D}$ is required.

Discrete vorticity equation. Next we define the vorticity ζ . This must be done weakly by introducing a test function to Equation (5) and integrating by parts, since $d(\mathbf{u} \cdot d\mathbf{x}_\Omega)$ cannot be evaluated in strong form. We obtain $\zeta \in \hat{\Lambda}^0(\Omega)$ from the solution of

$$\int_{\Omega} \gamma \wedge \star \zeta = - \int_{\Omega} d\gamma \wedge \star \mathbf{u} \lrcorner dS, \quad \forall \gamma \in \hat{\Lambda}^0(\Omega).$$

Since $d\gamma \in \hat{\Lambda}^1(\Omega)$ for arbitrary $\gamma \in \hat{\Lambda}^2(\Omega)$, we may select $\mathbf{w} \lrcorner dS = -d\gamma$ in Equation (7) to obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \gamma \wedge \star \zeta &= - \frac{d}{dt} \int_{\Omega} d\gamma \wedge \star \mathbf{u} \lrcorner dS \\ &= \int_{\Omega} d\gamma \wedge \mathbf{Q} \lrcorner dS + \int_{\Omega} \underbrace{d^2 \gamma}_{=0} \wedge (g(D-b) + K) dS, \\ &= \int_{\Omega} d\gamma \wedge \mathbf{Q} \lrcorner dS, \quad \forall \gamma \in \hat{\Lambda}^0(\Omega). \end{aligned}$$

This is the standard continuous finite element discretisation of the vorticity transport equation. Conservation of vorticity is a direct consequence of this, upon choosing $\gamma = 1$:

$$\frac{d}{dt} \int_{\Omega} \zeta dS = - \int_{\Omega} \underbrace{d(1)}_{=0} \wedge \mathbf{Q} \lrcorner dS = 0.$$

Discrete potential vorticity. Having diagnosed a vorticity, we can diagnose a potential vorticity $q \in \hat{\Lambda}^0(\Omega)$ from

$$\int_{\Omega} \gamma \wedge qD dS = \int_{\Omega} \gamma \wedge (\zeta + f) dS.$$

Then, the discrete potential vorticity satisfies

$$\frac{d}{dt} \int_{\Omega} \gamma \wedge qD dS + \int_{\Omega} d\gamma \wedge \mathbf{Q} \lrcorner dS = 0. \quad (9)$$

It remains to define \mathbf{Q} so that the diagnosed potential vorticity advection is mass-consistent, stable and oscillation-free. We shall follow identical steps to those taken in [RTKS10], namely we shall first diagnose a mass flux \mathbf{F} from Equation (8), and then shall insert $\mathbf{Q} = \mathbf{F}q$ in Equation (9), before making appropriate modifications to stabilise the advection.

Discrete mass flux. To compute the mass flux, the crucial observation is that since d maps from $\hat{\Lambda}^1(\Omega)$ onto $\hat{\Lambda}^2(\Omega)$, there must exist a discrete mass flux $\mathbf{F} \lrcorner dS \in \hat{\Lambda}^1(\Omega)$ such that

$$\frac{\partial}{\partial t} Dds = -d(\mathbf{F} \lrcorner dS). \quad (10)$$

It turns out that \mathbf{F} can be calculated locally, *i.e.* independently for each element without solving a global elliptic problem. Substitution of equation (10) into equation (8) and selection of $\phi dS = dS$ gives

$$\int_e d(\mathbf{F} \lrcorner dS) = \int_{\partial e} \mathbf{F} \lrcorner dS = \int_{\partial e} \mathbf{u} \tilde{D} \lrcorner dS,$$

which is satisfied if we require that

$$\int_{\partial e} \phi \, dS \wedge \star \mathbf{F} \lrcorner dS = \int_{\partial e} \phi \, dS \wedge \star \mathbf{u} \tilde{D} \lrcorner dS = 0, \quad \forall \phi \in \hat{\Lambda}^2.$$

The remaining degrees of freedom for \mathbf{F} can be fixed by

$$\int_e d\star(\phi \, dS) \wedge \mathbf{F} \lrcorner dS = - \int_e d\star(\phi \, dS) \wedge \mathbf{u} \lrcorner D \, dS + \int_{\partial e} \star(\phi \, dS) \wedge \mathbf{u} \lrcorner \tilde{D} \, dS, \quad \forall \phi \in \hat{\Lambda}^2,$$

plus local gauge conditions that do not alter $d\mathbf{u} \lrcorner dS$. This is essentially the Fortin projection into $\hat{\Lambda}^1(\Omega)$ [For77].

This calculation can be extended to the time-discretised case in which a slope limiter is applied before each timestep or Runge-Kutta stage [CS01]. In each element E , slope limiters aim to preserve monotonicity by adjusting $D \, dS$ in each element E in such a way that \bar{D}_E is preserved, where

$$\bar{D}_E = \frac{\int_E D \, dS}{\int_E dS}.$$

We write the action of the slope limiter on $D \, dS$ as

$$S(D \, dS) = D \, dS + d(\mathbf{F}_s \lrcorner dS).$$

Since the slope limiter preserves the mean value of $D \, dS$, we obtain

$$\int_{\partial E} \mathbf{F}_s \lrcorner dS = 0,$$

which means that we can set

$$\int_e \phi \mathbf{F}_s \lrcorner dS, \quad \forall \phi \in \hat{\Lambda}^2(\Omega),$$

for all edges $e \in \partial E$, and solve a problem for \mathbf{F}_s in the interior.

Discrete vorticity flux. We now choose $\mathbf{Q} = \mathbf{F}q$ in Equation (9), where \mathbf{F} is the discrete mass flux defined above, and we obtain

$$\frac{d}{dt} \int_{\Omega} \gamma q D \, dS - \int_{\Omega} d\gamma \wedge \mathbf{F}q \lrcorner dS = 0, \quad \forall \gamma \in \hat{\Lambda}^0. \quad (11)$$

Note that since γ and q are continuous, and \mathbf{F} has continuous normal components, we may integrate by parts in the discrete equations to obtain

$$\frac{d}{dt} \int_{\Omega} \gamma q D \, dS + \int_{\Omega} \gamma \wedge d(\mathbf{F}q \lrcorner dS) = 0, \quad \forall \gamma \in \hat{\Lambda}^0.$$

We can now verify the consistency property, namely that constant q remains constant, by choosing $q = 1$ in the above equation, obtaining

$$\int_{\Omega} \gamma D \frac{\partial q}{\partial t} \, dS + \int_{\Omega} \gamma \wedge \underbrace{\left(\frac{\partial}{\partial t} D \, dS + d(\mathbf{F} \lrcorner dS) \right)}_{=0} = 0, \quad \forall \gamma \in \hat{\Lambda}^0 \implies \frac{\partial q}{\partial t} = 0.$$

After computing \mathbf{Q} , we can insert it into the velocity equation and we obtain prognostic velocity evolution that is consistent with this diagnostic PV flux.

Stabilised vorticity flux. As previously discussed, Equation (11) is the continuous finite element approximation to the potential vorticity transport equation and thus requires stabilisation. This can be done by introducing diffusive fluxes into $\mathbf{Q} \lrcorner dS$ of an appropriate form, provided that the consistency property is preserved. The most basic type of stabilisation is from Laplacian diffusion that results from a down-gradient PV flux. Equation (11) then becomes

$$\frac{d}{dt} \int_{\Omega} \gamma q D dS - \int_{\Omega} d\gamma \wedge (\mathbf{F} \lrcorner dS + \star \kappa d q) = 0, \quad \forall \gamma \in \hat{\Lambda}^0,$$

where κ is the diffusivity, and we obtain $\mathbf{Q} \lrcorner dS = q\mathbf{F} \lrcorner dS - \star \kappa d q$. Note that this is mass-consistent since the diffusive flux vanishes for constant q .

A more sophisticated approach is provided by the streamline-upwind Petrov-Galerkin method [BH82] which replaces the test function γ with $\gamma + \tau \mathbf{u} \cdot d\gamma$, where τ is a suitable scalar coefficient that depends on \mathbf{u} and the mesh size h ; this effectively biases the test function in the upwind direction. Substitution into Equation (11) gives

$$\frac{d}{dt} \int_{\Omega} \gamma q D dS - \int_{\Omega} d\gamma \wedge \mathbf{F} \lrcorner dS + \int_{\Omega} d\gamma \wedge \tau \mathbf{u} \lrcorner \left(\frac{\partial}{\partial t} (q D dS) + q \mathbf{F} \lrcorner dS \right) = 0, \quad \forall \gamma \in \hat{\Lambda}^0,$$

and we obtain

$$\mathbf{Q} \lrcorner dS = q\mathbf{F} \lrcorner dS + \tau \mathbf{u} \lrcorner \left(\frac{\partial}{\partial t} (q D dS) + q \mathbf{F} \lrcorner dS \right).$$

As shown in [BH82], the Petrov-Galerkin method is stable and converges at the optimum rate since it is a residual-based modification of the equations. It is still possible for overshoots and undershoots to appear near to unresolved fronts, but these can be controlled by including a discontinuity capturing term [HM86], which consists of an additional Laplacian diffusion with spatially varying κ which is non-zero only in regions where the solution is poorly resolved, in which case the flux becomes

$$\mathbf{Q} \lrcorner dS = q\mathbf{F} \lrcorner dS + \tau \mathbf{u} \lrcorner \left(\frac{\partial}{\partial t} (q D dS) + q \mathbf{F} \lrcorner dS \right) - \star \kappa d q.$$

4. Primal-dual grid finite element formulation

In this section we provide an alternative formulation that makes use of a second set of spaces defined on a dual grid based on the $\mathbf{u} \cdot d\mathbf{x}_{\Omega}$ identification of vector fields with 1-forms. The idea is that when we want to apply ∇^{\perp} and $\nabla \cdot$ operators strongly we use the primal grid spaces as defined in the previous section, and when we want to apply ∇ and $\nabla^{\perp \cdot}$ strongly we use the dual grid spaces. This requires defining mappings between the primal and dual spaces which are defined *via* the Hodge star operator.

We start by selecting a set of finite element differential form spaces $\hat{\Lambda}_p^k \subset \Lambda^k$ and $\hat{\Lambda}_d^k(\Omega) \subset \Lambda^k$ on the primal grid and the dual grid respectively, satisfying

$$\begin{aligned} \hat{\Lambda}_p^0(\Omega) &\xrightarrow{d} \hat{\Lambda}_p^1(\Omega) \xrightarrow{d} \hat{\Lambda}_p^2(\Omega) \\ \hat{\Lambda}_d^0(\Omega) &\xrightarrow{d} \hat{\Lambda}_d^1(\Omega) \xrightarrow{d} \hat{\Lambda}_d^2(\Omega). \end{aligned}$$

We shall calculate with mass $D dS$ in $\hat{\Lambda}_p^2(\Omega)$, vorticity ζdS in $\hat{\Lambda}_d^2(\Omega)$, which are both discontinuous finite element spaces where discontinuous Galerkin/finite volume methods can be

used. We shall use the flux 1-form representation of velocity $\mathbf{u} \lrcorner dS$ in $\hat{\Lambda}_p^1(\Omega)$ (to evaluate divergence) and the circulation 1-form representation of velocity $\mathbf{v} \cdot d\mathbf{x}_\Omega$ (to evaluate vorticity) in $\hat{\Lambda}_p^2(\Omega)$.

Projection operators. Following [Hip01a, Hip01b], we define Hodge star projections Π from $\hat{\Lambda}_d(\Omega)^k$ to $\hat{\Lambda}_p^{2-k}(\Omega)$ given by

$$\int_{\Omega} \gamma \wedge \star \Pi \omega = \int_{\Omega} \gamma \wedge \omega, \quad \forall \gamma \in \hat{\Lambda}_p^k(\Omega),$$

and require that the dual spaces are chosen such that Π is invertible. This requirement somewhat limits the choice of spaces, the main options arising from finite element spaces that have the same structure as the primal/dual C-grid staggerings, for example [Chr08] suggests the hexagonal P1-RT0-P0 spaces for the primal mesh, and the triangular P1-N0-P0 spaces for the dual mesh, which would have the same structure as the hexagonal C-grid.

Primal and dual δ operator. The operator δ provides weak approximations to ∇ and ∇^\perp in the primal space, as described in the previous section for the primal finite element formulation, and weak approximations to ∇^\perp and $\nabla \cdot$ in the dual space. We shall now see that the key to the formulation is that we can define a simple relationship *via* the projection Π between d in the primal space and δ in the dual space, and *vice versa*.

Mapping from d to δ . For $\omega \in \hat{\Lambda}_d^k(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} \gamma \wedge \star \Pi d\omega &= \int_{\Omega} \gamma \wedge d\omega \\ &= - \int_{\Omega} d\gamma \wedge \omega \\ &= - \int_{\Omega} d\gamma \wedge \star \Pi \omega \\ &= \int_{\Omega} \gamma \wedge \star \delta \Pi \omega, \quad \forall \gamma \in \hat{\Lambda}_p^k(\Omega), \end{aligned}$$

where we may integrate by parts in the second step since $d\gamma$ and $d\omega$ are both well-defined. Hence, $\Pi d\omega = \delta \Pi \omega$, and the following diagram commutes:

$$\begin{array}{ccccc} & \xrightarrow{\delta} & & \xrightarrow{\delta} & \\ \hat{\Lambda}_d^2(\Omega) & \xleftarrow{d} & \hat{\Lambda}_d^1(\Omega) & \xleftarrow{d} & \hat{\Lambda}_d^0(\Omega) \\ \downarrow \Pi & & \downarrow \Pi & & \downarrow \Pi \\ \hat{\Lambda}_p^0(\Omega) & \xrightarrow{d} & \hat{\Lambda}_p^1(\Omega) & \xrightarrow{d} & \hat{\Lambda}_p^2(\Omega) \\ & \xleftarrow{\delta} & & \xleftarrow{\delta} & \end{array}$$

For example, this means that we may start with $\mathbf{u} \lrcorner dS \in \hat{\Lambda}_p^1(\Omega)$, and either obtain the primal vorticity $\zeta_p \in \hat{\Lambda}_p^0(\Omega)$ by directly applying δ , or by inverting Π to get $\mathbf{v} \cdot d\mathbf{x}_\Omega \in \hat{\Lambda}_d^1(\Omega)$, applying d to get the dual vorticity $\zeta_d dS \in \hat{\Lambda}_d^2(\Omega)$, and then projecting back to $\hat{\Lambda}_p^0(\Omega)$ with Π , *i.e.*,

$$\delta(\mathbf{u} \lrcorner dS) = \Pi d \Pi^{-1}(\mathbf{u} \lrcorner dS).$$

Prognostic equations. Within this framework, we retain the same equation for D on the primal grid, and modify the Coriolis term in the \mathbf{u} equation as follows:

$$\begin{aligned} \frac{d}{dt} \mathbf{u} \lrcorner dS + \int_{\Omega} \mathbf{w} \lrcorner dS \wedge \Pi \mathbf{Q} \cdot d\mathbf{x}_{\Omega} - \int_{\Omega} d(\mathbf{w} \lrcorner dS) \wedge \star (g(D-b) + K) dS &= 0, \quad \forall \mathbf{w} \lrcorner dS \in \hat{\Lambda}_p^1(\Omega), \\ \frac{d}{dt} \int_e \phi dS \wedge \star (D ds) - \int_e d \star (\phi dS) \wedge \mathbf{u} \lrcorner D dS + \int_{\partial e} \star (\phi dS) \wedge \mathbf{u} \lrcorner \tilde{D} dS &= 0, \quad \forall \phi dS \in \hat{\Lambda}_p^2(\Omega), \end{aligned}$$

with $\mathbf{Q} \cdot d\mathbf{x}_{\Omega} \in \hat{\Lambda}_d^1(\Omega)$. In our shorthand notation, we can rewrite the velocity equation as

$$\frac{\partial}{\partial t} \mathbf{u} \lrcorner dS + \Pi \mathbf{Q} \cdot d\mathbf{x}_{\Omega} + \delta (g(D-b) + K) = 0. \quad (12)$$

Primal vorticity. As before, the primal grid vorticity ζ_p is defined by $\zeta_p = \delta \mathbf{u} \lrcorner dS$, and application of δ to Equation (12) gives

$$\frac{\partial}{\partial t} \delta \mathbf{u} \lrcorner dS + \delta \Pi \mathbf{Q} \cdot d\mathbf{x}_{\Omega} = 0,$$

since $\delta^2 = 0$, and hence

$$\frac{\partial}{\partial t} \zeta_p + \delta \Pi \mathbf{Q} \cdot d\mathbf{x}_{\Omega} = 0. \quad (13)$$

Dual vorticity. We now introduce an equivalent dual grid vorticity $\zeta_d dS \in \hat{\Lambda}_0^2$ given by $\Pi \zeta_d dS = \zeta_p$. Substituting $\mathbf{u} \lrcorner dS = \Pi \mathbf{v} \cdot d\mathbf{x}_{\Omega}$ with $\mathbf{v} \cdot d\mathbf{x}_{\Omega} \in \hat{\Lambda}_d^1(\Omega)$, we obtain

$$\Pi \zeta_d dS = \zeta_p = \delta \mathbf{u} \lrcorner dS = \delta \Pi \mathbf{v} \cdot d\mathbf{x}_{\Omega} = \Pi d(\mathbf{v} \cdot d\mathbf{x}_{\Omega}),$$

and hence $\zeta_d = d(\mathbf{v} \cdot d\mathbf{x}_{\Omega})$ by invertibility of Π . Substitution into Equation (13) and application of the commutation relations for Π gives

$$\frac{\partial}{\partial t} \Pi \zeta_d + \Pi d \mathbf{Q} \cdot d\mathbf{x}_{\Omega} = 0,$$

and hence

$$\frac{\partial}{\partial t} \zeta_d + d \mathbf{Q} \cdot d\mathbf{x}_{\Omega} = 0,$$

by invertibility of Π . It remains to determine a suitable \mathbf{Q} .

Dual potential vorticity. This enables us to define a potential vorticity q with $q dS \in \hat{\Lambda}_d^2(\Omega)$, given by

$$\int_{\Omega} \phi dS \wedge \star (q D dS) = \int_{\Omega} \phi dS \wedge \star (\zeta_d + f) dS, \quad \forall \phi dS \in \hat{\Lambda}_d^2(\Omega).$$

We then write down the potential vorticity equation on dual cell e_d :

$$\frac{d}{dt} \int_{e_d} \phi q D dS - \int_{e_d} d\phi \wedge \mathbf{F} \lrcorner q dS + \int_{\partial e_d} \phi \wedge \tilde{q} \mathbf{F} \lrcorner dS = 0, \quad \forall \phi \in \hat{\Lambda}_d^2(\Omega),$$

noting that if q is constant then we obtain

$$\int_{e_d} \phi \frac{\partial q}{\partial t} D dS = - \int_{e_d} \phi q \wedge \underbrace{\left(\frac{\partial}{\partial t} D dS + d \mathbf{F} \lrcorner dS \right)}_{=0}, \quad \forall \phi \in \hat{\Lambda}_d^2(\Omega),$$

and hence

$$\frac{\partial q}{\partial t} = 0,$$

and we have mass consistency.

Dual vorticity flux. Identical to the mass flux reconstruction on the primal grid, we can then find $\mathbf{Q} \in \hat{\Lambda}_d^1(\Omega)$ such that

$$\frac{\partial}{\partial t} \zeta_d \, dS + d(\mathbf{Q} \cdot d\mathbf{x}_\Omega) = 0,$$

and hence

$$\frac{\partial \zeta_p}{\partial t} = \frac{\partial}{\partial t} \Pi \zeta_d \, dS = -\Pi d(\mathbf{Q} \cdot d\mathbf{x}_\Omega) = -\delta \Pi(\mathbf{Q} \cdot d\mathbf{x}_\Omega)$$

which means that

$$\frac{\partial}{\partial t} \delta \mathbf{u} \lrcorner dS + \delta \Pi(\mathbf{Q} \cdot d\mathbf{x}_\Omega) = 0,$$

which is consistent with the primal velocity equation

$$\frac{\partial}{\partial t} \mathbf{u} \lrcorner dS + \Pi(\mathbf{Q} \cdot d\mathbf{x}_\Omega) - \delta(gD + K) \, dS = 0.$$

5. Summary and outlook

In this paper, we used the finite element exterior calculus framework to develop two formulations for the shallow-water equations, a primal formulation that is defined on a single mesh where divergence is defined in the strong form but vorticity must be evaluated weakly using integration by parts, and a primal-dual formulation that makes use of a dual mesh where vorticity can also be computed in the strong form. Both of these formulations address the list of desirable properties given in the introduction, and they additionally have a conserved diagnostic potential vorticity. Both formulations provide a way to control oscillations in the divergence-free component of the velocity field (the component that dominates in large scale balanced flow in the atmosphere) by ensuring that the potential vorticity remains mass-consistent and oscillation-free. In the primal-dual case this can be achieved since the potential vorticity is diagnosed on a discontinuous space where discontinuous Galerkin/finite volume methods can be used to provide stable shape-preserving potential vorticity fluxes. In the primal case, the potential vorticity is computed in a continuous finite element space, but streamline-upwind Petrov-Galerkin methods with discontinuity capturing are compatible with the framework and can be used to provide stable potential vorticity fluxes with minimal oscillations.

This work is part of the UK GungHo Dynamical Core project, which is a NERC/STFC collaboration between UK academics and the UK Met Office to design a dynamical core for the Unified Model that will perform well on the next generation of massively parallel supercomputers. In Phase 1 of the project, one of the main goals is to determine the horizontal discretisation that will be used, with the shallow-water equations on the sphere providing an environment to investigate this. The aim is to develop discretisations on a pseudouniform

grid¹ that have all of the desirable properties listed in the introduction, whilst maintaining the accuracy of the current model. Numerical accuracy is crucial since it reduces grid imprinting (structure in the numerical errors that reflects the structure of the grid, *e.g.* larger errors near the corners of a cubed sphere). This work opens up a number of possibilities that could be sufficiently accurate for operational use. In [CS12] it was shown that to avoid spurious mode branches it is necessary to select finite element spaces that have a 2:1 ratio of velocity DOFs to pressure DOFs, which suggests the BDFM1 space on triangles with an icosahedral mesh in the primal formulation or RT0 on quadrilaterals with a cubed sphere mesh in the primal-dual formulation. There is an argument to be made that spurious Rossby mode branches arising from increasing velocity DOFs relative to this ratio are not harmful since they have very low frequencies and will just be passively advected by the flow. This suggests the BDM1 space on triangles in the primal formulation or the RT0 space on hexagons in the primal-dual formulation, which both have a 3:1 ratio. The next steps in this work are to analyse the numerical convergence and dispersion relations of all of these schemes and to benchmark them against the usual suites of testcases and against solutions from the Unified Model formulation.

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¹A grid for which the ratio of smallest to largest edge lengths remains bounded as the maximum edge length tends to zero.

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